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LONGITUDINAL MANPOWER PLANNING MODELS

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LONGITUDINAL MANPOWER PLANNING MODELS

by

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AUGUST 1973 ORC 73-15

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ABSTRACT

Several manpower planning models are presented that exploit the longitudinal stability of manpower cohorts. The manpower planning process is described along with the problem of identifying and obtaining various types of longitudinal data. An infinite horizon linear program for calculating minimum cost manpower input plans is found to have a straightforward solution in a great many cases and to yield an easily implemented approximation technique in other cases.

1. INTRODUCTION

This paper formulates manpower planning models for a system consisting of many skill categories; the particular application is to the enlisted force in the U.S. Navy. Interactive computer models of these theoretical formulations have been developed and implemented to aid decision-makers who wish to test the effects of alternative manpower policies on staffing requirements and future manpower budgets. These interactive manpower planning models have a variety of uses, to:

(1) predict the manpower requirements that will be fulfilled by the current stock of manpower.

(2) calculate unfulfilled requirements and the new inputs necessary to meet them.

(3) identify bottlenecks in the manpower planning process.

(4) assist in preparation of future manpower budgets.

(5) simulate the effects of manpower policy changes on future manpower needs.

(6) relate alternate personnel retention and performance assumptions to the need for future inputs.

(7) calculate minimum cost input schedules when lower bounds on requirements are given.

Details of the interactive positions of the models are not included here since they are to a large extent dependent on the particular hardware/software combination used. Rather we emphasize the underlying mathematical structure.

Individuals in any skill category can be identified by several characteristics: examples are; rank, salary, number of years of experience in the skill category, length of service in the Navy, and personal attributes such as age and measures of performance. The models presented in this paper are designed to assist in preparing manpower budgets and meeting aggregate strength requirements. For these purposes we have chosen to identify individuals in the enlisted force according to skill category and length of service (LOS) in the Navy.

Section 2 of the paper describes the underlying manpower flow models. The models are based on the assumption of longitudinal stability in the service lifetimes of different manpower cohorts. We show that the accession schedule that exactly meets requirements is found by solving a set of lower triangular system of linear equations. Section 3 relates several methods of describing the longitudinal behavior of manpower cohorts and shows how the flow model's parameters can be estimated from existing Navy data. In Section 4 we present an infinite horizon linear program for the calculation of future manpower inputs and strength levels with the objective of minimizing discounted costs. We derive readily verifiable conditions on the inputs to the infinite horizon problem that guarantee that the equality solution, described in Section 2, will be optimal. In cases where the equality solution is not optimal we describe a finite linear program that will approximate solutions of the infinite horizon program.

The models presented in this paper examine the relationships between three factors: (i) the current manpower situation as described by the LOS distributions of different skill ratings, (ii) the survivor fractions that will determine the longitudinal behavior of manpower cohorts, and (iii) the manpower requirements for future times. The size of our models and the type of calculations performed allow policy makers to quickly analyze the impact of various assumptions and policies.

2. THE UNDERLYING COHORT FLOW MODELS

We consider an organization which is divided into many skill categories where movement between categories typically involves retraining of the individual. In this paper we consider each skill category separately; a subsequent paper, will discuss transfers and other interactions between skill categories.

We idealize the evolution of the skill category by analyzing its changes at discrete points in time $(i = \ldots -2,-1,0,1,2, \ldots)$. We say that period i is the interval between times $(i - 1)$ and i ; it is a future period if $i > 1$, a past period if $i < 0$, and the current period if $i = 1$. In period i a number of people x_i enter the skill category; that group is called *cohort* i and x_i is the number of *accessions* in cohort i . Let α_{ij} be the fraction of the cohort entering in period i which is still present and available to meet requirements at time $i + j$ $(j \geq 0)$. Let z_k be the requirement in the skill category at time k and let $(m + 1)$ be the maximum number of periods an individual is allowed to stay in the system. Thus $\alpha_{ij} = 0$ if $j > m$. For some future time k we have

(1) $$z_k = x_k\alpha_{k,0} + x_{k-1}\alpha_{k-1,1} + \ldots + x_{k-m}\alpha_{k-m,m} .$$

Equation (1) simply says that the requirement at time k is made up of fractions of cohorts which survive from earlier periods. Thus it is natural to call the α_{ij}'s the *survivor fractions* for the cohort which enters at time i .

At time 0 , the history of past accessions is given by the vector $(x_{-m},x_{1-m}, \ldots, x_{-2},x_{-1},x_0)$. The current inventory of people is given by $x_0\alpha_{0,0} + x_{-1}\alpha_{-1,1} + \ldots + x_{-m}\alpha_{-m,m}$, and contains the remaining fractions of past inputs. This quantity is called the *current legacy*, say y_0 . In future period k the legacy y_k from past inputs up to and including period 0 will be

$$y_k = x_0\alpha_{0,k} + x_{-1}\alpha_{-1,k+1} + \dots + x_{k-m,}\alpha_{-m,k+m}\ , \quad \text{if} \quad k \le m\ ,$$

(2)

$$= 0 \qquad\qquad \text{if} \quad k > m\ .$$

Suppose we have a planning horizon of T periods with requirements $z_1, z_2, \dots z_T$. From equations (1) and (2) we see that future cohort sizes must satisfy†

(3)

$$\begin{aligned}
&\alpha_{1,0}x1 && = z_1 - y_1 \\
&\alpha_{1,1}x_1 + \alpha_{2,0}x_2 && = z_2 - y_2 \\
&\dots \\
&\dots \\
&\alpha_{1,T-1}x_1 + \alpha_{2,T-2}x_2 + \dots + \alpha_{T,0}x_T && = z_T - y_T\ . \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

Here we have assumed

A1: requirements are met exactly.

Under A1 it is quite possible that for a given set of z_k's , y_k's and α_{ij}'s some x_k could be negative. Such a result would say that in order to exactly meet requirements in all periods 1,2, ..., T it will be necessary to remove people from the skill category in period k . In practice, this might be accomplished through retraining.

This section concentrates on the equality solution for several reasons. First, it is misleading to state the problem as if the accessions (the x_k) are the only variables which the decision-maker can influence. The legacies (y_k) , requirements (z_k) and to some extent the survivor fractions (α_{ij})

†Recall that $\alpha_{ij} = 0$ if $j > m$.

can all be changed or explicitly influenced by manpower policies. Second, plans that are eventually recommended will probably conform to the equality constraints since budget restrictions do not generally allow for slack in the system. Third, we intend to use the models in this section to test the effects of alternate policies on several objectives: (1) the departure of realistic requirements from ideal requirements, (2) the smoothing of stocks and accessions, (3) the impact of policy changes on personnel and finally (4) the retraining costs associated with switching personnel from one specialty to another. In Section 4 we shall drop A1 and treat the net requirements as lower bounds and look for cost minimizing accession schedules.

In the remainder of this paper we make an important second assumption

A2: the survivor fractions $\alpha_{i,j}$ are stationary from period to period. That is, $\alpha_{i,j} = \alpha_j$, independent of i and independent of x_i .

Under assumption A2 equation (3) to simplify to

$$
(4) \qquad
\begin{aligned}
&\alpha_0 x_1 &&= z_1 - y_1 \\
&\alpha_1 x_1 + \alpha_0 x_2 &&= z_2 - y_2 \\
&\cdots\cdots\cdots\cdots && \\
&\alpha_{T-1} x_1 + \alpha_{T-2} x_2 + \cdots + \alpha_0 x_T &&= z_T - y_T \, .
\end{aligned}
$$

In these equations the legacies are given by

$$
(5) \qquad
\begin{aligned}
&y_1 = \alpha_1 x_0 + \alpha_2 x_{-1} + \cdots\cdots\cdots + \alpha_m x_{1-m} \\
&y_2 = \alpha_2 x_0 + \alpha_3 x_{-1} + \cdots\cdots + \alpha_m x_{2-m} \\
&\cdots\cdots\cdots \\
&y_T = \alpha_T x_0 + \cdots + \alpha_m x_{T-m}
\end{aligned}
$$

Equation (4) can be used in a number of ways. We have mentioned already that given the requirements, legacies and survivor fractions, (4) can be used to calculate new cohort input requirements for each period of the planning horizon T . Alternatively, given planned inputs over the next T periods the z_i's can be considered as the result of these inputs. Also, given requirements and planned inputs, the legacies which satisfy equation (4) can be determined. These possibilities will be discussed further in the section on interactive programming.

To this point we have motivated equations (4) and (5) by considering z_i and y_i as stocks of people and x_i as a flow of people per year. It is not necessary to define α , x , y , and z in that manner. We could also speak of stocks and flows of money or effective personnel or use another unit of time. For example if c_j is the cost associated with an individual in the j^{th} year of service, the total cost in year i will be

$$(6) \qquad c_0\alpha_0 x_i + c_1\alpha_1 x_{i-1} \cdots + c_m\alpha_m x_{i-m} .$$

The cost legacy, that will be incurred in future period i due to current, period 0 , manpower legacies is

$$(7) \qquad c_i\alpha_i x_0 + c_{i+1}\alpha_{i+1} x_1 \cdots + c_m\alpha_m x_{i-m} .$$

The use of equations (4) and (5) depends on the availability of data to estimate the various parameters. The next section examines several possible data configurations and presents some actual examples of U.S. Navy enlisted force data.

3. DATA

To employ the cohort model described in Section 2, it is necessary to obtain information about past accessions x_i and some knowledge about past survivor fractions α_j. Typically, data which is available in most organizations is of the cross-sectional type. Thus we may know the number currently in a given skill category and the length of time each person has been in the category. In the case of the U.S. Navy enlisted force this type of cross-sectional information is available for a number of past planning periods. Data on the past inputs and the survivor fractions are not available, and so we must either eliminate these from our equations or find methods of determining them from available data.

We shall demonstrate below how *estimates* of the survivor fractions and past skill category accessions can be obtained from several different configurations, and present an example using U.S. Navy data. The reader should, however, not focus too strongly on the problem of estimating survivor fractions from historical data. Survivor fractions realized by past cohorts are the result of former manpower policies, and the cohort's behavioral characteristics. In the context of the U.S. Navy these data are influenced by early out policies, retraining and transfer decisions, changes in the length of tours, the mixture of regular and reserve forces, and the fraction of draft motivated accessions. Simple projections using historical estimates of survivor fractions will, at best, show the results of continuing past manpower policies. The specification of future survivor fractions is, to a great extent, the specification of future manpower policy.

Despite this warning on the use of historical estimates, we shall now show how these estimates can be made with several different data configurations. In the first case we assume that past accession levels x_i for $i \leq 0$, are known as well as past stocks z_j . Equation (1) of Section 1, combined with assumption A2 can be written

$$(1) \qquad z_j = x_j\alpha_0 + x_{j-1}\alpha_1 + x_{j-2}\alpha_2 \cdots + x_{j-m}\alpha_m \ .$$

If z_j is known for $j = 0,-1, \ldots, -m$ and x_j for $j = 0,-1, \ldots, -2m$, then system (1) represents $m + 1$ equations in the $m + 1$ unknowns $\alpha_0,\alpha_1, \ldots, \alpha_m$. However, it is highly unlikely that data on skill category inputs will be maintained at the precise degree of aggregation called for by the model over the past $2m$ periods. We might find the aggregate input into several skill levels, or a change in manpower accounting methods in the past $2m$ periods. Thus, in general, it is necessary to estimate the survivor fractions without knowledge of past accessions.

We are now left with the task of using the models described in Section 2 when past accessions are not known and cross-sectional data is available. We define n_{ij} to be the number of people in year i with j years of service. From the cohort assumption A2,

$$(2) \qquad n_{ij} = x_{i-j}\alpha_j \ .$$

An individual has j years of service if his length of service (LOS) lies in the interval $[j,j + 1)$. It does not follow that an individual with LOS equal to j has been in a skill category for j years. It is possible to be unrated for several years, and it is possible to be transferred between skill categories. In many of the critical highly trained ratings, however, the LOS and time in skill category coincide.

Note that formula (2) allows one to calculate the legacy without knowledge of the past accession levels. Given the current longitudinal data n_{0j} and the survivor fractions α_j that will obtain in future periods, formula (2) can be used to calculate the past accession levels that would be consistent with both current LOS data and the assumption that the survivor fractions α_j applied in past periods. These accession levels can then be used in equation (5) of Section 2 to calculate the legacy.

It remains to find some method of estimating the survivor fractions and of relating them to other measures of longitudinal stability namely: the continuation rate β_i , and the age fraction γ_{ij} . These measures will be defined, and then related to the survivor fractions α_i , and cross-sectional data n_{ij} .

The *continuation rate* β_j is defined as

$$\beta_j = \frac{n_{i,j}}{n_{i-1,j-1}} .$$

It is most convenient to interpret β_j as the fraction of those with $j - 1$ years of service in year $i - 1$, that remain (continue) in year i . It is possible, although this effect is not explicitly considered in our model, to enter a skill category with more than one year of service. Therefore historical estimates of β_j from reliable data can yield numbers that are greater than one. The notation has anticipated the result that β_j is independent of i due to assumption A2. From (2), we have

$$\text{(4)} \qquad \beta_j = \frac{x_{i-j}\alpha_j}{x_{i-1-(j-1)}\alpha_{j-1}} = \frac{\alpha_j}{\alpha_{j-1}} . \qquad j = 1,2, \ldots, m .$$

The value of $\beta_0 = 1$ is assumed. Thus we can calculate β from α using (4) and the reverse calculation is simply:

$$\text{(5)} \qquad \alpha_j = \alpha_0 \sum_{i=1}^{j} \beta_j \qquad j = 1,2, \ldots m .$$

The *age fraction* γ_{ij} is the fraction of people in year i that have j years of service. It follows that

$$\gamma_{ij} = \frac{n_{ij}}{z_i}$$

where $z_i = \sum_{j=0}^{m} n_{ij}$. When assumption A3 holds, then

$$\gamma_{ij} = \frac{x_{i-j}\alpha_j}{z_i} .$$

To be a useful measure of longitudinal stability, the γ_{ij} should be stationary, i.e. independent of i . However, stationary age fractions are in general incompatible with assumption A2. In particular, if A2 holds *and* γ_{ij} is independent of i , then one can easily determine that the system must be growing at a constant rate $(\theta - 1)$ with $x_i = \theta^i x_0$, $z_i = \theta^i z_0$, and

$$\gamma_{ij} = \frac{\alpha_j}{\theta^j z_0} .$$

We have seen that the survivor fractions α , and continuation rates β , are useful measures of longitudinal cohort stability, while the age fraction γ_{ij} does not isolate the effects of varying past levels of accessions from the behavior patterns of the cohorts. We conclude this discussion with a formula that shows how the quantities α , β , γ are calculated from the cross-sectional data. Since the α and β are essentially flow rates, at least two years of cross-sectional data are necessary to determine their values.

$$\beta_j = \frac{n_{ij}}{n_{i-1,j-1}} = \frac{z_i\gamma_{ij}}{z_{i-1}\gamma_{i-1,j-1}}$$

(6)

$$\alpha_j = \frac{n_{ij} \cdots n_{i2} \cdot n_{i1}}{n_{i-1,j-1} \cdots n_{i-1,1} \cdot n_{i-1,0}} \alpha_0 .$$

In what follows we shall demonstrate how these concepts can be applied to some actual data.

The flow models described in Section 1 were designed to assist in manpower planning for the U.S. Navy's enlisted ranks. That force comprises approximately 550,000 people in nine ranks and 60 skill categories which are commonly called ratings.

Tables 1 and 2 contain the length of service distributions of five ratings. If we take June 30, 1972 to be time 0, then Table 1 gives the values of $n_{-1,j}$ and Table 2 the values of $n_{0,j}$. The particular data presented in Tables 1 and 2 was abstracted from Rating Career Factor reports prepared by the Personnel System Research Department of the Naval Personnel Training and Research Laboratory, San Diego.

The Rating Career Factor reports contain the *tail* of the age fraction as well as the total inventory z_i in each rank. Let Γ_{ij} be the fraction in period i, that have completed j or more periods of service. Clearly

$$\Gamma_{i,j} = \gamma_{i,j} + \gamma_{i,j+1} \cdots + \gamma_{i,m}$$

and

$$\gamma_{ij} = \Gamma_{i,j} - \Gamma_{i,j+1}$$

Thus to find n_{ij} we must calculate

$$n_{ij} = z_i(\Gamma_{i,j} - \Gamma_{i,j+1}) .$$

For the Boatswain's Mate rating we have

$$n_{0,5} = z_{-1}(\Gamma_{-1,5} - \Gamma_{-1,6})$$

$$= (10973)(5.97 - 56.2) = 384 .$$

The original source of this data is the enlisted personnel master tape, that is maintained by the Management Information Division of the Bureau of Naval Personnel (Pers N). The data kept by Pers N is used on a daily basis for current management of the Navy; it is a necessity of the cross-sectional

TABLE 1

LENGTH OF SERVICE DISTRIBUTIONS AS OF 6/30/71

Length of Service	Ratings BM	QM	ET	CS	BT
0	225	1186	5044	1435	1488
1	1547	1078	3848	1372	1926
2	1513	911	3544	1173	2156
3	382	82	2305	357	198
4	438	141	1762	503	285
5	449	87	762	493	340
6	438	125	653	399	318
7	449	141	544	325	285
8	561	152	479	378	285
9	516	130	436	315	340
10	449	168	414	284	274
11	404	125	371	231	329
12	348	130	349	231	351
13	326	130	262	273	351
14	471	168	197	451	548
15	516	168	218	545	559
16	236	103	131	294	209
17	427	93	110	503	307
18	483	103	88	420	220
19	124	28	44	116	56
20	57	17	23	32	23
21	102	17	23	43	23
22	158	28	44	64	23
23	102	12	23	32	1
24	124	17	23	64	23
25	79	17	23	32	23
26	91	17	23	32	12
27	102	17	23	53	1
28	79	17	1	32	12
29	35	6	1	11	1

TABLE 2

LENGTH OF SERVICE DISTRIBUTIONS AS OF 6/30/72

Length of Service	Ratings BM	QM	ET	CS	BT
0	166	1317	3578	1649	2501
1	769	596	3578	830	961
2	2832	1050	3578	1489	2026
3	319	107	2996	331	286
4	341	62	2091	321	170
5	385	132	1121	470	254
6	418	87	647	480	317
7	418	117	518	361	275
8	418	122	454	301	254
9	484	137	432	351	254
10	495	127	410	311	317
11	429	157	389	291	254
12	396	117	367	231	317
13	330	132	324	231	339
14	319	127	238	261	349
15	462	167	195	441	539
16	495	167	216	530	560
17	231	97	130	291	201
18	297	62	87	311	170
19	253	56	23	191	96
20	78	16	23	71	33
21	45	11	23	21	12
22	67	16	23	31	22
23	122	21	44	41	12
24	78	11	23	31	12
25	100	6	23	41	12
26	67	16	23	21	22
27	78	11	1	41	1
28	89	21	23	41	12
29	23	6	1	11	1

type. That is, on any given day, their data file gives a current listing of Navy Personnel. We obtained values of $n_{-1,j}$ from report E1S-9H, Pers A12, Bureau of Naval Personnel, dated 9/30/71. These values conformed closely to the $n_{-1,j}$ calculated from the age tails. From Tables 1 and 2, we see that on June 30, 1971 there were 1513 people in the category "Boatswain's Mate" in the Navy with between 2 and 3 years of service. Similarly there were 371 "Electronics Technicians" with more than 11 and not more than 12 years of service.

Using this data and formula (6), we can calculate a point estimate of the continuation rates β_j . For example for Boatswain's Mates.

$$\beta_4 = \frac{n_{0,2}}{n_{-1,3}} = \frac{341}{382} = 0.89$$

on

$$\beta_2 = \frac{n_{0,2}}{n_{-1,1}} = \frac{2832}{1547} = 1.83 \ .$$

In this case, we see that the continuation rate exceeds 1. This is due to a large number of inputs into BM rating with two years of unrated service. The continuation rates and survivor fraction calculated for the skill categories BM and ET are presented in Tables 3.

In the critical skill, ET, the continuation rates are less than 1 and survivor fractions decrease. This follows from the Navy's practice of training ET's from recruits with a zero LOS. The BM rating draws its input from the unrated enlisted ranks, and generally with an LOS equal to 2. The BM survivor fractions can be modified by dividing them by $\alpha_2 = 6.28$. The modification allows us to interpret x_i as the number of accessions at time i that *eventually* enter the BM skill category, and $\alpha_j x_i$ as the number in the BM skill category in period $i + j$.

TABLE 3

LOS	CONTINUATION RATES BM	CONTINUATION RATES ET	SURVIVOR FRACTIONS BM	SURVIVOR FRACTIONS ET
0	1	1	1	1
1	3.43	0.71	3.43	0.71
2	1.83	0.93	6.28	0.66
3	0.21	0.85	1.32	0.56
4	0.89	0.91	1.18	0.51
5	0.88	0.64	1.04	0.32
6	0.33	0.85	0.97	0.27
7	0.95	0.79	0.92	0.22
8	0.93	0.83	0.85	0.18
9	0.86	0.90	0.74	0.16
10	0.96	0.94	0.70	0.15
11	0.96	0.94	0.68	0.14
12	0.98	0.99	0.66	0.14
13	0.95	0.93	0.63	0.13
14	0.98	0.91	0.62	0.12
15	0.98	0.99	0.60	0.12
16	0.96	0.99	0.57	0.12
17	0.98	0.99	0.57	0.12
18	0.70	0.79	0.39	0.09
19	0.52	0.25	0.20	0.02
20	0.62	0.50	0.12	0.01
21	0.78	0.99	0.10	0.01
22	0.65	0.99	0.07	0.01
23	0.77	0.99	0.05	0.01
24	0.76	0.99	0.04	0.01

4. MINIMIZING ACCESSION COSTS

The accession schedule that meets future manpower requirements z_t exactly is found by solving the equations

$$
\begin{array}{llr}
 & \alpha_0 x_0 & = z_1 - y_1 \\
 & \alpha_1\alpha_1 + \alpha_0 x_2 & = z_2 - y_2 \\
 & \vdots \qquad \vdots & \vdots \\
(1) & \alpha_{m-1}x_1 + \alpha_{m2}x_2 + \ldots + \alpha_0 x_m & = z_m - y_m \\
 & \alpha_m x_1 + \alpha_{m-1}x_2 + \ldots + \alpha_1 x_m + \alpha_0 x_{m+1} & = z_{m+1} \\
 & \qquad \alpha_m x_2 + \ldots + \alpha_2 x_m + \alpha_1 x_{m+1} + \alpha_0 x_{m+2} & = z_{m+2} \\
 & \qquad\qquad \vdots \qquad \vdots \qquad \vdots & \vdots
\end{array}
$$

In this section we shall relax the assumption that future manpower requirements are satisfied exactly. Instead we shall treat the variables z_t as lower bounds on the manpower level at time t . Moreover, we shall restrict the accessions x_t to be nonnegative. Thus the equalities in (1) will be replaced by inequalities $(\geq)$. This leaves us with an infinite system of linear inequalities that will, in general, have a large number of possible solutions. To obtain a single accession schedule in this case we can specify a performance criterion and then select the accession schedule that optimizes that criterion.

In the analysis that follows we assume that the performance criterion is to minimize the present worth of all future accession costs. This objective is obtained by discounting future costs to today's dollars and then summing over all future periods. We obtain the extremely simple and useful result that the optimal accession schedule is independent of the costs. In addition, we will show that there are many conditions of practical interest when the equality solution discussed in earlier sections is indeed an optimal solution to the infinite-horizon program. Finally, we will discuss several practical methods by which the infinite program can be truncated and approximated by a finite program with T rather than an infinite number of periods. The method of truncation that should be used depends on the conditions the user wishes to assume for planning periods in the distant future.

The Infinite Horizon Program

There are several approaches to the solution of (1) when the restriction of exactly meeting the manpower requirements is relaxed. In describing a cost minimization model in which the manpower requirements are considered to be lower bounds we assume that manpower requirements can in fact be modelled by such inequality restrictions.

The problem is to choose the nonnegative vector $(x_1, x_2, \ldots)$ that satisfies

$$
(2) \qquad \begin{array}{llllll}
\alpha_0 x_1 & & & \geq z_1 - y_1 & & \\
\alpha_1 x_1 & + \alpha_0 x_2 & & \geq z_2 - y_2 & & x_t \geq 0 \\
\alpha_2 x_1 & + \alpha_1 x_2 & + \alpha_0 x_3 & \geq z_3 - y_3 & & \\
\vdots & \vdots & \vdots & \vdots & &
\end{array}
$$

Note that it is possible to consider x_t as the number of accessions above some minimal level. Then z_t should be interpreted as the requirements left unsatisfied by the minimal accession schedule.

Let c_k for $k = 0,1,2, \ldots, m$ be the cost of training and support of an individual in the k^{th} year of service. The discounted cost of an accession is thus

$$(3) \qquad c = \sum_{j=0}^{m} (\alpha_j c_j)\delta^j = \alpha_0 c_0 + \alpha_1 c_1 \delta + \alpha_2 c_2 \delta^2 + \ldots + \alpha_m c_m \delta^m$$

where δ is a discount factor less than one. The total discounted cost of an accession schedule $(x_1, x_2, \ldots,)$ is therefore

$$(4) \qquad \sum_{j=1}^{\infty} c x_j \delta^{j-1} = c x_1 + \delta c x_2 + \delta^2 c x_3 + \ldots$$

Since an accession in any period j costs $\delta^{j-1}c$, *it is evident that the value of* c *will not affect the optimal solution.* In fact, we can, without any loss in generality, use

$$(5) \qquad \sum_{j=1}^{\infty} x_j \delta^{j-1} = x_1 + \delta x_2 + \delta^2 x_3 + \ldots$$

as the objective to be minimized subject to the constraints in (2).

The Dual Program and Optimality Conditions

The dual of the infinite linear program in (5) and (2) is to find non-negative variables $u_1, u_2, \ldots$ which

$$(6) \qquad \text{Maximize} \quad u_1(z_1 - y_1) + u_2(z_2 - y_2) + \ldots$$

subject to the inequality constraints

$$(7) \qquad \begin{array}{rl} u_1\alpha_0 + u_2\alpha_1 + u_3\alpha_2 + \ldots & \leq 1 \\ u_2\alpha_0 + u_3\alpha_1 + \ldots & \leq \delta \qquad u_t \geq 0 \\ u_3\alpha_0 + \ldots & \leq \delta^2 \\ \vdots \quad & \ \ \vdots \end{array}$$

A feasible solution of the dual problem always exists and is given by $u_t = \mu\delta^{t-1}$, where $\mu = 1/\left(\sum_{j=0}^{m} \alpha_j \delta^j\right)$. This solution is strictly positive and satisfies each dual constraint as an equality.

The main results of linear programming do not carry over to infinite linear programs; the strong duality theorem can fail, [1], and in some cases even the weak duality theorem and the related sufficiency condition (complementary slackness) can fail, [2]. Fortunately, our problem has a great deal of special structure and under two reasonable assumptions we shall establish a useful lower bound on the optimal value of the infinite problem and determine that the solution of (1) is optimal if it is nonnegative, i.e. feasible.

The first assumption is that $\alpha_0 > 0$ and that $\alpha_j \geq 0$ for all j. The second assumption is on the m^{th} order difference equation $\sum_{j=0}^{m} \alpha_j x_{t-j} = z_t$, with initial conditions given by $(x_0, x_{-1}, \ldots, x_{1-m})$. It is obvious from equation (5) in Section 2 that the initial conditions are also specified by the legacy $y_1, y_2, \ldots, y_m$, and that the unique solution of this difference equation is simply the solution of (1) in this section. The difference equation is called *asymptotically stable* if all the complex roots of its characteristic polynomial $\sum_{j=0}^{m} \alpha_j w^{m-j}$, have absolute value strictly less than one. In particular, the condition is satisfied if: $1 \geq \alpha_j \geq \alpha_{j+1}$ and $\alpha_{i-1} > \alpha_i > \alpha_{i+1}$ for some i, and appears to be satisfied by other survivor fraction data that we have encountered to date. In, [3], Chapter 10 it is demonstrated that asymptotic stability implies that for any initial conditions and any bounded z_t the solution of the difference equation will be bounded. Let $\tilde{x}_t$ be the unique, bounded solution of (1). From our two assumptions it follows that $x_t = \text{Max}\ (0, \tilde{x}_t)$ is a bounded and feasible solution for the constraints (2).

To establish optimality conditions it is useful to examine the T period approximations of the infinite problem. The T period problem is

$$\text{Minimize} \sum_{j=1}^{T} \delta^{j-1} x_j$$

(8)
$$\text{Subject to} \sum_{j=0}^{t-1} \alpha_j x_{t-j} \geq z_t - y_t$$

$$x_t \geq 0 \quad \text{for} \quad t = 1,2, \ldots, T$$

Notice that any solution of the infinite problem will have its first T components $(x_1, x_2, \ldots, x_T)$ feasible for the T period approximation. Moreover, the solution $u_t = \mu\delta^{t-1}$ for $t = 1,2, \ldots, T$ will be feasible for the dual of problem (8). Thus by the weak duality theorem for problem (8), we have

$$\sum_{t=1}^{T} \delta^{t-1} x_t \geq \sum_{t=1}^{T} \mu\delta^{t-1}(z_t - y_t)$$

for all T and any solution of the infinite problem. This gives us a lower bound $\sum_{t=1}^{\infty} \mu\delta^{t-1}(z_t - y_t)$ on the optimal value of the infinite problem. Finally, since $\delta < 1$, and $\tilde{x}_t$ is bounded it is possible to show that $\sum_{t=1}^{\infty} \delta^{t-1}\tilde{x}_t = \sum_{t=1}^{\infty} \mu\delta^{t-1}(z_t - y_t)$. Thus the possibly infeasible solution $\tilde{x}_t$ achieves the lower bound. It follows that $\tilde{x}_t$ is optimal if it is feasible, i.e. if each element of $\tilde{x}_t$ is nonnegative.

In summary the result of this section is that under the hypothesis of asymptotic stability, boundedness of the requirements sequence z_t, and nonnegativity of the survivor fractions, the equality solution $\tilde{x}_t$ is optimal if it is nonnegative.

Optimality of Equality Solution

Intuition suggests the equality solution of (2) will often be an optimal solution of the infinite horizon planning problem posed in (5). Identification of the conditions which must exist for the equality solution to be nonnegative and thus optimal gives considerable insight into the structure of the problem. These conditions place bounds on the magnitude of allowable changes in future manpower requirements. Moreover, optimality of the equality solution gives us an analytic expression for total system cost as a function of the continuation rates, net requirements, and costs.

To begin the analysis of this section we look for solution of a reduced system of linear equations

$$\begin{aligned}
x_1 \qquad\qquad\qquad\qquad &= 1 \\
\beta_1 x_1 + x_2 \qquad\qquad &= \phi_1 \\
\beta_1\beta_2 x_1 + \beta_1 x_2 + x_3 \qquad &= \phi_1\phi_2 \\
\beta_1\beta_2\beta_3 x_1 + \beta_1\beta_2 x_2 + \beta_1 x_3 + x_4 &= \phi_1\phi_2\phi_3 \\
\vdots \qquad\qquad\qquad\qquad & \quad \vdots
\end{aligned} \tag{9}$$

which is equivalent to normalizing the equality system in (1) by the constant $\frac{z_1 - y_1}{\alpha_0}$. In (9) ϕ_j is defined to be the ratio of successive net requirements, i.e. $z_{j+1} - y_{j+1}/z_j - y_j$. Alternatively, one can view (9) as the equality system in (2) with $\alpha_0 = 1$, $\beta_1\beta_2 \ldots \beta_j = \alpha_j$, $z_1 - y_1 = 1$ and $(z_{j+1} - y_{j+1}) = \phi_1\phi_2 \ldots \phi_j$. In either case, multiplication of a solution vector $x = (x_1, x_2, \ldots)$ of (9) by $z_1 - y_1/\alpha_0$ yields the equality solution of (1). It should be noted by the reader that just as the continuation rates $\beta_1, \beta_2, \ldots \beta_j$ measure the period to period increases or decreases in the fraction surviving an additional period, the growth rates $\phi_1, \phi_2, \ldots \phi_j$ measure the period to period increases or decreases in the net requirements that must be met by new accessions.

With this definition of terms it is tempting to believe that so long as

$\phi_j \geq \beta_j$, i.e. the one period increase (decrease) in net requirements exceeds the one period change in the continuation rates, nonnegative accessions will meet new requirements in the next period. However, the simple numerical example having data

$$(\phi_1,\phi_2,\phi_3,\phi_4) = (5 \quad 0.2 \quad 1 \quad 1)$$

(10a)

$$(\beta_1,\beta_2,\beta_3,\beta_4) = (2 \quad 0.05 \quad 1 \quad 1)$$

yields a solution

(10b) $$(x_1,x_2,x_3,x_4,x_5) = (1 \quad 3 \quad {}^{-}5.1 \quad 10.8 \quad {}^{-}20.49)$$

with negative components. This example shows that the condition $\phi_j - \beta_j \geq 0$ is insufficient to guarantee nonnegative accessions in all periods if there are periods of very large requirements followed by a period of small requirements.

It is not difficult to show by a direct substitution of unknowns in the first three equations of (9) that an equality solution x_j satisfies

$$x_1 = 1$$

(11) $$x_2 = \phi_1 - \beta_1$$

$$x_3 = (\phi_1\phi_2 - \beta_1\beta_2) - \beta_1(\phi_1 - \beta_1)$$

$$\vdots$$

By a reversion of the series in (9) it can be shown that in general x_j satisfies the j^{th} order linear, homogeneous difference equation,

(12) $$x_{j+1} = \sum_{i=1}^{j} x_{j+1-i}(\phi_j - \beta_i) \prod_{k=0}^{i-1} \beta_k \; . \qquad j \geq 1$$

With (12) it is now possible to obtain x_{j+1} recursively in terms of $x_1, x_2, \ldots x_j$. As we have already pointed out the solution of x_j only depends on β_i's and ϕ_i's with $i \leq j$. In other words changing the requirements on periods beyond j does not affect the accessions in time periods on or before j. Since continuation rates are nonnegative it is simple to show that the sequence of inequalities

$$(13) \qquad \phi_j \geq \text{Max } \{\beta_1, \beta_2, \ldots \beta_j\} \qquad j \geq 1$$

is sufficient to ensure that $x_j \geq 0$. The system of inequalities in (13) is, of course, much more restrictive than the one period inequalities $\phi_j \geq \beta_j$ (all j) as it compares the growth rate in one period to *all previous* continuation rates. The proof is straightforward: $x_1 = 1$ is nonnegative and by (11) $\phi_1 \geq \beta_1$ also implies $x_2 \geq 0$. If we now assume that $x_1, x_2, \ldots x_j$ are all nonnegative and that (13) holds, then x_{j+1} in (12) is a sum of nonnegative terms. Thus, by induction on n we see that (13) is sufficient to guarantee that all accessions x_j are nonnegative.

Again we use the data in (10a) to indicate why the equality solution in (10b) fails to be optimal. Notice that

$$(14) \qquad \begin{aligned} \phi_1 &= 5 \geq \text{Max } \{2\} = 2 \\ \phi_2 &= 0.2 < \text{Max } \{2, 0.05\} = 2 \\ \phi_3 &= 1 < \text{Max } \{2, 0.05, 1\} = 2 \\ \phi_4 &= 1 < \text{Max } \{2, 0.05, 1, 1\} = 2 . \end{aligned}$$

Necessary conditions for the x_j in (12) to be nonnegative are obtained by noting that we can always rewrite the $(j + 1)^{st}$ equation in (9) as

$$(15) \qquad \phi_1\phi_2\phi_3 \ldots \phi_j - \beta_1\beta_2\beta_3 \ldots \beta_j = \sum_{i=2}^{j+1} x_i \prod_{k=0}^{j+1-i} \beta_k .$$

Since the right hand side is nonnegative if all $x_i \geq 0$ it follows that the inequalities on cumulative products,

$$\phi_1 \geq \beta_1$$

$$\text{(16)} \qquad \phi_1\phi_2 \geq \beta_1\beta_2$$

$$\phi_1\phi_2\phi_3 \geq \beta_1\beta_2\beta_3 \text{ , etc.}$$

must hold. While $\phi_j \geq \beta_j$ imply (16) the converse is not true as it is quite possible that $\phi_j < \beta_j$ while $\phi_1\phi_2 \cdots \phi_j \geq \beta_1\beta_2 \cdots \beta_j$. Thus the simple and local test of whether the growth rate in net requirements exceeds the continuation rate in *each period* lies somewhere between the necessary conditions of (16) and the more global sufficiency conditions in (13).

When the equality solution is nonnegative the optimal dual variables for problem (7) are given by $\mu\delta^{t-1}$. However, to obtain useful total and marginal cost information it is necessary to reintroduce (3), the discounted cost per accession. The total cost of an optimal solution is

$$\text{(17)} \qquad \left(\sum_{j=0}^{m} (\alpha_j c_j)\delta^j \Big/ \sum_{j=0}^{m} \alpha_j \delta^j\right) \sum_{t=1}^{\infty} (z_t - y_t)\delta^{t-1} \text{ .}$$

This formula has a reasonable interpretation in the case where $\alpha_0 = 1$, and the α_j are nonincreasing. Let $\alpha_j - \alpha_{j+1}$ be the probability that an individual's lifetime, i.e. maximum LOS, is equal to j . With this stochastic interpretation of the survivor fractions we define two random variables: T the individual's lifetime and K the total support cost of the individual. When δ is equal to 1, the term in brackets in (17) is simply $E[K]/E[T]$. Thus we can save by keeping $E[T]$ fixed and reducing costs. Notice, however, that if we attempt to increase expected lifetime by changing the α_j , then the cost will change also. We can get a more accurate estimate of the impact of possible changes by rewriting the first term of (17) with $\delta = 1$, and α_j expressed in terms of continuation rates

$$\left(\sum_{j=0}^{m} c_j \prod_{k=0}^{j} \beta_k\right) \Big/ \sum_{j=0}^{m} \prod_{k=0}^{j} \beta_k$$

The derivative of (18) with respect to β_ℓ is

$$\frac{\sum_{j=\ell}^{m} \left\{c_j - E[K]/E[T]\right\}\alpha_j}{\beta_\ell E[T]}$$

This expression reinforces some intuitive feelings about the system. If the cost in the periods following ℓ is greater than average, then an increase in β_ℓ will surely increase costs. On the other hand, if all of the downstream costs are less than average, then it will reduce costs when β_ℓ is increased.

Special Cases

This section examines several special cases and derives tighter and more easily verified conditions under which the equality solution is optimal.

First if the β_j are nondecreasing, then $\phi_j \geq \beta_j$ implies $\phi_j \geq \beta_i$ for $i \leq j$, thus the conditions $\phi_j \geq \beta_j$ are necessary and sufficient for the equality solution to be nonnegative.

In a second case, if the α_j are nonincreasing, then $\beta_j \leq 1$ for all j . Moreover, nonincreasing α_j imply, see equation 5 Section 2, that the legacy y_t is nonincreasing. If z_t is nondecreasing, it follows that $z_t - y_t$ is nondecreasing and thus that $\phi_j \geq 1$ for all j . Therefore we have optimality of the equality solution under the readily verified conditions z_t nondecreasing and α_j nonincreasing.

If we further specialize the first case so that $\beta_j = \beta < \phi_j$ for all j , then we can write

(19) $$x_{j+1} = \phi_{j-1}\left(\frac{\phi_j - \beta}{\phi_{j-1} - \beta}\right) x_j \qquad j = 2,3, \ldots$$

with $x_1 = 1$, and $x_2 = \phi_1 - \beta$. If $\phi_j = \phi$ $j = 1,2, \ldots, T$, and $\phi_j = 1$ thereafter we obtain

(20) $$x_{j+1} = \begin{Bmatrix} \phi^{j-1}(\phi - \beta) & j \leq T \\ \phi^T(1 - \beta) & j > T \end{Bmatrix}$$

<u>Calculating Approximately Optimal Accessions Schedules</u>

It is often not possible to determine if the equality solution of (1) is nonnegative. These situations typically involve relatively large decreases in requirements in the first few periods together with a large legacy. In these cases we must find some procedure for either solving or approximating the solution of the infinite horizon problem (2) and (5).

This section will briefly describe three methods for calculating approximately optimal solution to the infinite horizon optimization problem. Each of the three methods is based on a partition of the original infinite problem into a T period finite problem followed by an infinite problem that commences at time $T + 1$. The hope is that the system will settle down enough so that the problem starting at time $T + 1$ will have a nonnegative equality solution regardless of the choice of $(x_1, x_2, \ldots, x_T)$.

The first method simply ignores the decisions and constraints for time $T + 1$ onwards. Thus we are solving problem (8) of this section. This procedure is quite simple, however, it can lead to optimal programs that save in periods 1 through T by presenting difficult initial conditions for the second problem that commences at time $T + 1$. Since the problem that starts at time $T + 1$ is not explicitly considered in the objective, there is no penalty to deter this type of behavior.

The second method attempts to provide a smooth transition to equilibrium by fixing accessions at their equilibrium value for periods $T + 1$ onward. The assumption is that $z_T = z_t$ for $t \geq T$, and that $x_t = z_T / \sum_{j=0}^{m} \alpha_j$ for all $t \geq T$. Thus the accessions in periods $1 - m, 2 - m, \ldots, -1, 0$ and $T + 1, T + 2, \ldots$ are all known. We must determine the accessions in periods 1 through T in order to satisfy the lower bound requirement in the first $T + m$ time periods. This leads to a linear program with $T + m$ inequality constraints and T nonnegative variables $x_1, \ldots, x_T$. The dual linear program, with T inequality constraints and $T + m$ nonnegative variables, is easier to solve. Unfortunately this truncation procedure has not been effective in numerical examples we have solved to date. We frequently obtain relatively low values of x_{T-1}, x_{T-2}, etc. and a relatively large value of x_T. In effect, the program satisfies the boundary restriction by making a last period correction. This behavior is contrary to the smooth transition to equilibrium that the model was designed to produce.

The third method is based on the theory developed previously on optimality conditions. The lower bound on the optimal value of the infinite horizon problem that starts at time $T + 1$, must consider the additional legacy due to the accessions in periods 1 through T. The bound is

$$\sum_{j=T+1}^{\infty} \mu\delta^{j-1}\left(z_j - y_j - \sum_{k=1}^{T} \alpha_{j-k} x_k\right).$$

If we add the value of the first T periods, $\sum_{k=1}^{T} \delta^{k-1} x_k$, and rearrange terms we obtain a lower bound for the original infinite problem in terms of the decision variables $x_1, \ldots, x_T$.

$$\sum_{k=1}^{T}\left(\delta^{k-1} - \delta^{T}\mu \sum_{\ell=1}^{m+k-T} \delta^{\ell-1}\alpha_{\ell+T-k}\right)x_k - \sum_{j=T+1}^{\infty} \mu\delta^{j-1}(z_j - y_j) .$$

Notice that the expression on the right is a constant, independent of the decisions in period 1 through T . Thus the linear program we solve is

$$\text{Minimize} \quad \sum_{k=1}^{T}\left(\delta^{k-1} - \delta^{T}\mu \sum_{\ell=1}^{m+k-T} \delta^{\ell-1}\alpha_{\ell+T-k}\right)x_k$$

$$\text{(21)} \qquad \text{Subject to} \quad \sum_{k=1}^{t} \alpha_{t-k}x_k \geq z_t - y_t$$

$$x_t \geq 0 \qquad t = 1,2, \ldots, T .$$

Notice that the third method is similar to the first, except for the objective which explicitly contains a penalty cost on the legacy created for the infinite problem starting at time $T + 1$.

In our calculations we have used the third method. There are two main reasons for this: first, the lower bound is exact when the equality solution is optimal for the problem commencing in period $T + 1$; second, the calculations we have made using actual survivor fractions have led to realistic answers.

The dual linear program is stated below:

$$\text{Maximize} \quad \sum_{t=1}^{T} u_t(z_t - y_t)$$

$$\text{(22)} \qquad \text{Subject to} \quad \sum_{j=0}^{T-t} \alpha_j u_{t+j} + v_t = \delta^{t-1} - \delta^{T}\mu \sum_{\ell=0}^{m+t-T} \delta^{\ell-1}\alpha_{\ell+t-t}$$

$$u_t \geq 0 \ , \ v_t \geq 0 \quad \text{for} \quad t = 1,2, \ldots, T$$

The interpretation of the dual variables is straight forward: u_t is the marginal change in the optimal cost of increasing net requirements $(z_t - y_t)$, and v_t is the marginal change in the optimal cost associated with increasing the lower bound on accessions in period t . The dual constraints can be interpreted as the equations $\sum_{j=0}^{m} \alpha_j u_{t+j} + v_t = \delta^{t-1}$ where u_t and v_t for $t \geq T$ have been assigned the values $\delta^{t-1}\mu$ and 0 .

The example below was solved for the rating ET, electronics technician, using the survivor fraction and age distribution data contained in Tables 2 and 3 of Section 3. We imposed a lower bound of 1750 accessions per year, a discount factor $\delta = 0.95$; and $z_t = 16000$ for $t \geq 6$.

	1	2	3	4	5
z_t	20000	18000	16000	16000	16000
x_t	2112	1750	1750	2098	2828
$\sum_{j=0}^{m} \alpha_j x_{t-j}$	20000	18363	16922	16000	16000
u_t	0.5	0.0	0.0	0.19	0.18
v_t	0.0	0.35	0.2	0.0	0.0

Since z_t is constant for $t \geq 6$ and the survivor fractions α_j are nonincreasing for the ET rating we know that the equality solution is optimal for the problem beginning in period 6 regardless of the accessions in periods 1 through 5. Thus the solution above is optimal for the infinite horizon problem.

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